*Journal of Sound and Vibration* (2000) **231**(2), 277–290
doi:10.1006/jsvi.1999.2503, available online at http://www.idealibrary.com on IDEAL®

# ANALYSIS OF THE EQUIVALENT CIRCUIT OF PIEZOELECTRIC CERAMIC DISK RESONATORS IN COUPLED VIBRATION

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(*Received* 21 *April* 1999, *and in final form* 19 *July* 1999)

Based on the piezoelectric effect and the wave equation, the coupled vibration of the piezoelectric ceramic thick disk resonator is studied when the shearing and torsion are neglected. The coupled vibration of the disk resonator is reduced to two equivalent vibrations, one being the equivalent radial vibration, and the other the equivalent longitudinal vibration. The relation between these two equivalent extensional vibrations is analyzed and the two-dimensional equivalent circuit of the thick disk resonator is derived. Compared with one-dimensional theory, an additional force is produced in the two-dimensional equivalent circuit. It is obvious that this force results from the coupling between the radial and longitudinal vibrations in the thick disk. It is shown theoretically that the resonance frequency of the thick disk in coupled vibration can be computed in an analytical method, and the measured resonance frequencies are in good agreement with the computed results.



## 1. INTRODUCTION

Piezoelectric ceramic devices are widely used as transducers in applications from telephone speakers to sonar arrays. In many traditional applications, the piezoelectric ceramic materials can be manufactured in a variety of geometrical shapes and have been used as ultrasonic transducers, filters, oscillators and transmitters. In these applications, the fundamental vibrational modes are thickness extensional vibrational mode, plane radial vibrational mode, and length extensional vibrational mode and shear vibrational mode [1]. For these vibrational modes, traditional analysis theory is one-dimensional, and therefore, certain condition is required for the geometrical shapes of the devices. For example, for the plane radial vibration and the thickness extensional vibration, the thickness of the resonator must be much smaller than its radius; while for the longitudinal vibration of a piezoelectric ceramic rod, its length must be much larger than the radius. However, in practical cases, the geometrical dimensions of the devices are limited, the vibration of the piezoelectric ceramic resonator is not an ideal one-dimensional vibration, and the coupling between the longitudinal and the lateral vibrations must be considered. In this case, one-dimensional analysis theory should not be

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used. The solution to this problem is that numerical methods should be used, or the traditional one-dimensional theory may be improved.

To analyze the vibrational characteristics of piezoelectric ceramic resonators of finite dimensions, numerical methods such as the finite element method have been employed [4–8]. However, to model the resonator in coupled vibration, the main drawback is generally the large size of the numerical problem. In our previous work, the coupled vibration of the piezoelectric ceramic circular resonator was analyzed using an approximate numerical method [9]. In this paper, the coupled vibration of the piezoelectric ceramic disk resonator is studied using a kind of improved analytical method and the equivalent circuit is derived. The basic concepts of this improved analytical method are as follows. When a piezoelectric ceramic disk resonator of finite dimension is excited by an external electric field, the coupled vibration of the resonator is assumed to be divided into two equivalent vibrations, one being the extensional vibration in the thickness direction, the other the radial vibration in the radial direction. However, these two equivalent vibrations are not independent of each other; they are correlated by a mechanical coupling coefficient. In this method, since the direction of the external exciting electric field is parallel to the polarization direction of the piezoelectric ceramic circular disk, the shearing strain and torsion in the resonator are ignored. By this method, the equivalent circuit of the disk resonator in coupled vibration is analyzed, and the resonance frequency equations are derived based on the developed equivalent circuit.

## 2. ANALYSIS OF THE EQUIVALENT CIRCUITS OF THE DISK RESONATOR IN COUPLED VIBRATION

A geometrical diagram of the piezoelectric ceramic disk resonator with electrodes on its two end surfaces is shown in Figure 1. The thickness and radius of the resonator are $l$ and $a$. When an external alternating electric field is applied to the resonator in the thickness direction, the resonator will vibrate. In this case, since the direction of the external exciting electric field is parallel to the polarization direction of the piezoelectric ceramic circular disk, the vibration is mainly a coupled one of two extensional vibrations; the torsion and shearing strain may be ignored. In Figure 1, $F_{31}$, $v_{31}$, $F_{32}$, $v_{32}$, $F_{ra}$, $v_{ra}$ are the external forces and velocities on its two ends and the side. In cylindrical co-ordinates, the piezoelectric and wave equations for the axial symmetrical coupled vibration of the resonator are as follows:

$$S_r = s_{11}^E T_r + s_{12}^E T_\theta + s_{13}^E T_z + d_{31} E_3, \tag{1}$$

$$S_\theta = s_{12}^E T_r + s_{11}^E T_\theta + s_{13}^E T_z + d_{31} E_3, \tag{2}$$

$$S_z = s_{13}^E (T_r + T_\theta) + s_{33}^E T_z + d_{33} E_3, \tag{3}$$

$$D_3 = d_{31}(T_r + T_\theta) + d_{33} T_z + \varepsilon_{33}^T E_3, \tag{4}$$

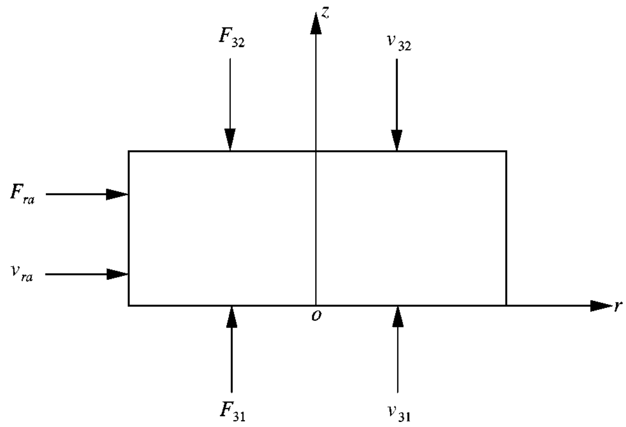


Figure 1. A geometrical diagram of the piezoelectric ceramic disk resonator in coupled vibration.

$$\rho \frac{\partial^2 U_r}{\partial t^2} = \frac{\partial T_r}{\partial r} + \frac{T_r - T_\theta}{r}, \tag{5}$$

$$\rho \frac{\partial^2 U_z}{\partial t^2} = \frac{\partial T_z}{\partial z}. \tag{6}$$

Here, $S_r$, $S_\theta$, $S_z$ and $T_r$, $T_\theta$, $T_z$ are the strains and stresses in the radial, tangential and longitudinal directions, $s_{ij}(i, j = 1, 2, 3)$ the elastic complaint coefficients in a constant electric field, $d_{31}$ and $d_{33}$ the piezoelectric constants, $E_3$ and $D_3$ the electric field intensity and the electric flux density, $\varepsilon_{33}^{\mathrm{T}}$ the dielectric constant, $\rho$ the density of the ceramic material, and $U_r$, $U_z$ the radial and longitudinal displacements. Since the edge effect of the electric field and the shearing strain and stress are ignored, $T_{rz}$, $T_{r\theta}$, $T_{\theta z}$, $S_{rz}$, $S_{r\theta}$, $S_{\theta z}$, $E_1$, $E_2$, $D_1$, $D_2$ can be ignored, and the expressions of $\partial D_3/\partial z = 0$ and $\partial E_3/\partial r = 0$ are applicable to this case. On the other hand, as the extensional vibration in the resonator is predominant, the shear and torsion can be ignored, and therefore, the tangential displacement $U_\theta$ is also ignored. The relation between the strain and displacement is

$$S_r = \frac{\partial U_r}{\partial r}, \qquad S_\theta = \frac{U_r}{r}, \qquad S_z = \frac{\partial U_z}{\partial z}. \tag{7}$$

### 2.1. EQUIVALENT CIRCUIT OF THE EQUIVALENT RADIAL VIBRATION OF THE RESONATOR IN COUPLED VIBRATION

From equations (1) and (2), the following can be obtained:

$$S_r - S_\theta = (s_{11}^E - s_{12}^E)(T_r - T_\theta), \tag{8}$$

$$S_\theta + S_r = (s_{11}^E + s_{12}^E)(T_r + T_\theta) + 2s_{13}^E T_z + 2d_{31} E_3. \tag{9}$$

Let $n = T_z/(T_r + T_\theta)$; $n$ is defined as the mechanical coupling coefficient between different vibrational modes. Equation (9) can be rewritten as

$$S_r + S_\theta = (s_{11}^E + s_{12}^E + 2s_{13}^E n)(T_r + T_\theta) + 2d_{31}E_3. \tag{10}$$

From equation (8) and (10), we have

$$T_r + T_\theta = \frac{S_r + S_\theta - 2d_{31}E_3}{s_{11}^E + s_{12}^E + 2s_{13}^E n}, \tag{11}$$

$$T_r = \frac{1}{2}\left(\frac{S_r - S_\theta}{s_{11}^E - s_{12}^E} + \frac{S_r + S_\theta - 2d_{31}E_3}{s_{11}^E + s_{12}^E + 2s_{13}^E n}\right). \tag{12}$$

Substituting the expressions of $T_r$ and $(T_r - T_\theta)$ into equation (5) yields

$$\rho\frac{\partial^2 U_r}{\partial t^2} = E_r\left(\frac{\partial^2 U_r}{\partial r^2} + \frac{1}{r}\frac{\partial U_r}{\partial r} - \frac{U_r}{r^2}\right). \tag{13}$$

Here,

$$E_r = \frac{1}{2}\left(\frac{1}{s_{11}^E - s_{12}^E} + \frac{1}{s_{11}^E + s_{12}^E + 2s_{13}^E n}\right) = \frac{s_{11}^E + s_{13}^E n}{(s_{11}^E - s_{12}^E)(s_{11}^E + s_{12}^E + 2s_{13}^E n)}$$

is called the equivalent elastic constant in radial vibration. For harmonic vibration, $U_r = U_{ra}\exp(\mathrm{j}\omega t)$. The solution to equation (13) is

$$U_{ra} = A_r J_1(k_r r) + B_r Y_1(k_r r). \tag{14}$$

Here, $k_r = \omega/V_r$, $V_r = (E_r/\rho)^{1/2}$, and $k_r$ and $V_r$ are the equivalent wavenumber and sound speed in radial vibration. $J_1(k_r r)$ and $Y_1(k_r r)$ are Bessel functions. When $r = 0$, $Y_1(k_r r)$ is divergent. Therefore, the radial displacement function for the disk resonator is

$$U_{ra} = A_r J_1(k_r r). \tag{15}$$

The radial velocity is

$$v_r = \partial U_r/\partial t = \mathrm{j}\omega U_r = \mathrm{j}\omega A_r J_1(k_r r)\exp(\mathrm{j}\omega t). \tag{16}$$

From Figure 1, the radial velocity of the resonator on the side is $v_{ra}$, and $A_r$ can be obtained:

$$A_r = -v_{ra}\exp(-\mathrm{j}\omega t)/\mathrm{j}\omega J_1(k_r a). \tag{17}$$

Let $F_{ra}$ be the external force on the side of the resonator; we have

$$F_{ra} = -2\pi a l T_r(r = a). \tag{18}$$

Substituting the expression of $T_r$ into equation (18), the following can be obtained:

$$F_{ra} = \frac{2\pi a l d_{31} E_3}{s_{11}^E + s_{12}^E + 2s_{13}^E n} + \frac{2\pi a l E_r k_r}{j\omega} \times \left[\frac{J_0(k_r a)}{J_1(k_r a)} - \frac{1}{k_r a} \times \frac{1 - \nu_{12} - 2\nu_{13} n}{1 - \nu_{13} n}\right] v_{ra}. \tag{19}$$

Here, $\nu_{12} = -s_{12}^E/s_{11}^E$, $\nu_{13} = -s_{13}^E/s_{11}^E$. For the equivalent radial vibration of the resonator, the voltage and current of the resonator are as follows:

$$V_{31} = \int_0^l E_3 \, dz = E_3 l, \tag{20}$$

$$I_{31} = j2\pi\omega \int_0^a D_3 r \, dr. \tag{21}$$

From equation (4) we have

$$D_3 = (d_{31} + n d_{33})(T_r + T_\theta) + \varepsilon_{33}^T E_3. \tag{22}$$

Substituting equation (11) into equation (22) and combining equation (17) yield

$$D_3 = \varepsilon_{33}^T E_3 + \frac{(d_{31} + n d_{33})}{s_{11}^E + s_{12}^E + 2s_{13}^E n}\left[A_r k_r J_0(k_r r) - 2d_{31} E_3\right]. \tag{23}$$

Substituting equation (23) into equation (22) and combining equation (7) yield

$$I_{31} = \frac{j\omega\pi a^2 \varepsilon_{33}^T}{l} \times \left[1 - \frac{2d_{31}(d_{31} + n d_{33})}{(s_{11}^E + s_{12}^E + 2s_{13}^E n)\varepsilon_{33}^T}\right] \times V_{31} - \frac{2\pi a(d_{31} + n d_{33})}{s_{11}^E + s_{12}^E + 2s_{13}^E n} \times v_{ra}. \tag{24}$$

Let

$$C_{0r} = \frac{\pi a^2 \varepsilon_{33}^T}{l} \times \left[1 - \frac{2d_{31}(d_{31} + n d_{33})}{(s_{11}^E + s_{12}^E + 2s_{13}^E n)\varepsilon_{33}^T}\right], \quad N_{31} = \frac{2\pi a(d_{31} + n d_{33})}{s_{11}^E + s_{12}^E + 2s_{13}^E n},$$

$C_{0r}$ and $N_{31}$ are defined as the equivalent clamped capacitance and electromechanical conversion coefficient in coupled vibration, equation (24) can be rewritten as

$$I_{31} = j\omega C_{0r} V_{31} - N_{31} v_{ra}. \tag{25}$$

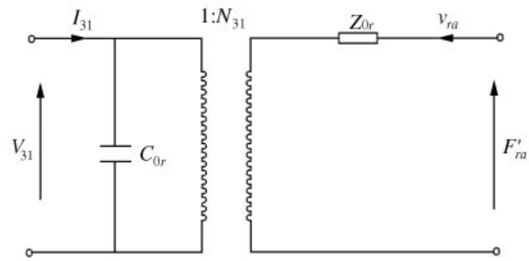


Figure 2. The equivalent circuit of the equivalent radial vibration of the resonator in coupled vibration.

Substituting equation (20) and the expression of $N_{31}$ into equation (19) yields

$$F_{ra} = Z_{0r} v_{ra} + N_{31} V_{31} - \frac{2\pi n a d_{33} V_{31}}{s_{11}^E + s_{12}^E + 2 s_{13}^E n}, \tag{26}$$

where

$$Z_{0r} = \frac{\rho S_r V_r}{j} \times \left[ \frac{J_0(k_r a)}{J_1(k_r a)} - \frac{1 - v_{12} - 2v_{13} n}{k_r a(1 - v_{13} n)} \right], \quad \text{and} \quad S_r = 2\pi a l$$

is the side area of the resonator. Let $F'_{ra} = F_{ra} + 2\pi n a d_{33} V_{31}/(s_{11}^E + s_{12}^E + 2s_{13}^E n)$; equation (26) can be rewritten as

$$F'_{ra} = Z_{0r} v_{ra} + N_{31} V_{31}. \tag{27}$$

From equations (25) and (27), the radial equivalent circuit of the resonator in coupled vibration can be obtained as shown in Figure 2. Compared with the traditional equivalent circuit of the thin piezoelectric ceramic disk in plane radial vibration, the difference is that for the coupled vibration, an additional force is created in the equivalent circuit. It is obvious that when the thickness of the resonator is much smaller than the radius, the mechanical coupling coefficient is very small, and the additional force can be ignored. The equivalent circuit of the resonator in coupled vibration can be reduced to the traditional one.

### 2.2. THE EQUIVALENT CIRCUIT OF THE EQUIVALENT THICKNESS VIBRATION OF THE RESONATOR IN COUPLED VIBRATION

From equations (3) and (4), we have

$$S_z = (s_{33}^E + s_{13}^E/n) T_z + d_{33} E_3, \tag{28}$$

$$E_3 = [D_3 - (d_{33} + d_{31}/n) T_z]/\varepsilon_{33}^T. \tag{29}$$

Substituting equation (29) into (28) yields

$$T_z = E_z(S_z - d_{33}D_3/\varepsilon_{33}^{\mathrm{T}}). \tag{30}$$

Here $E_z = [s_{33}^E + (s_{13}^E/n) - (d_{33}/\varepsilon_{33}^{\mathrm{T}})(d_{33} + (d_{31}/n))]^{-1}$ is called the equivalent longitudinal elastic constant of the disk resonator in coupled vibration. Let $\nu_{31} = -s_{13}^E/s_{33}^E, \lambda_{31} = -d_{31}/d_{33}$. $E_z$ can be rewritten as

$$E_z = \frac{1}{s_{33}^E[1 - \nu_{31}/n - K_{33}^2(1 - \lambda_{31}/n)]}, \tag{31}$$

where $K_{33}^2 = d_{33}^2/(s_{33}^E\varepsilon_{33}^{\mathrm{T}})$, and $K_{33}$ is the electro-mechanical coupling coefficient of the slender piezoelectric ceramic rod in longitudinal vibration. Substituting equation (30) into equation (6) yields

$$\rho\frac{\partial^2 U_z}{\partial t^2} = E_z\frac{\partial^2 U_z}{\partial z^2}. \tag{32}$$

Substituting $U_z = U_{za}\exp(\mathrm{j}\omega t)$ into equation (32) yields

$$\frac{\mathrm{d}^2 U_{za}}{\mathrm{d}z^2} + k_z^2 U_{za} = 0. \tag{33}$$

Here $k_z = \omega/V_z$, $V_z = (E_z/\rho)^{1/2}$, $k_z$ and $V_z$ are the equivalent longitudinal wave number and sound speed of the disk resonator. The solution to equation (33) is

$$U_{za} = A_z\sin(k_z z) + B_z\cos(k_z z). \tag{34}$$

From equation (34), the velocity of the resonator in thickness direction can be derived:

$$v_z = \partial U_z/\partial t = \mathrm{j}\omega U_z = \mathrm{j}\omega[A_z\sin(k_z z) + B_z\cos(k_z z)]\exp(\mathrm{j}\omega t). \tag{35}$$

From Figure 1, using equation (35), we have

$$v_{31} = v_z(z = 0) = \mathrm{j}\omega B_z\exp(\mathrm{j}\omega t), \tag{36}$$

$$v_{32} = -v_z(z = l) = -\mathrm{j}\omega[A_z\sin(k_z l) + B_z\cos(k_z l)]\exp(\mathrm{j}\omega t). \tag{37}$$

From the above equations, the constants can be obtained:

$$A_z = -\frac{1}{\mathrm{j}\omega}\times\left[\frac{v_{31}}{\tan(k_z l)} + \frac{v_{32}}{\sin(k_z l)}\right]\exp(-\mathrm{j}\omega t), \tag{38}$$

$$B_z = \frac{v_{31}}{\mathrm{j}\omega}\exp(-\mathrm{j}\omega t). \tag{39}$$

Substituting equations (38) and (39) into equation (30) yields

$$T_z = -\frac{k_z E_z}{j\omega} \times \left[ \left( \frac{v_{31}}{\tan k_z l} + \frac{v_{32}}{\sin k_z l} \right) \cos(k_z z) + v_{31} \sin(k_z z) \right] - \frac{d_{33} D_3 E_z}{\varepsilon_{33}^{\mathrm{T}}}. \quad (40)$$

From Figure 1, the external forces can be expressed as

$$F_{31} = -\pi a^2 T_z(z=0) = \frac{k_z E_z \pi a^2}{j\omega} \times \left( \frac{v_{31}}{\tan k_z l} + \frac{v_{32}}{\sin k_z l} \right) + \frac{d_{33} D_3 E_z \pi a^2}{\varepsilon_{33}^{\mathrm{T}}}, \quad (41)$$

$$F_{32} = -\pi a^2 T_z(z=l) = \frac{k_z E_z \pi a^2}{j\omega} \times \left( \frac{v_{31}}{\sin k_z l} + \frac{v_{32}}{\tan k_z l} \right) + \frac{d_{33} D_3 E_z \pi a^2}{\varepsilon_{33}^{\mathrm{T}}}, \quad (42)$$

The voltage and current of the resonator in thickness vibration can be obtained:

$$I_{33} = j\omega D_3 \pi a^2, \quad (43)$$

$$V_{33} = \int_0^l E_3 \, dz. \quad (44)$$

Substituting equation (40) into equation (29) and then into equation (44) yields

$$V_{33} = \frac{1}{\varepsilon_{33}^{\mathrm{T}}} \times \left[ 1 + \left( d_{33} + \frac{d_{31}}{n} \right) \frac{d_{33} E_z}{\varepsilon_{33}^{\mathrm{T}}} \right] D_3 l + \frac{(d_{33} + d_{31}/n) E_z}{j\omega \varepsilon_{33}^{\mathrm{T}}} (v_{31} + v_{32}). \quad (45)$$

Substituting equation (43) into equation (45) yields

$$I_{33} = j\omega C_{0z} V_{33} - N_{33}(v_{31} + v_{32}). \quad (46)$$

Here

$$C_{0z} = \frac{\pi a^2 \varepsilon_{33}^{\mathrm{T}}}{l[1 + (d_{33} + d_{31}/n) d_{33} E_z / \varepsilon_{33}^{\mathrm{T}}]}, C_{0z}$$

is called the equivalent clamped capacitance,

$$N_{33} = \frac{(d_{33} + d_{31}/n) E_z \pi a^2}{l[1 + (d_{33} + d_{31}/n) d_{33} E_z / \varepsilon_{33}^{\mathrm{T}}]} = \frac{(d_{33} + d_{31}/n) \pi a^2}{l(s_{33}^E + s_{31}^E/n)}, N_{33}$$

is defined as the equivalent electro-mechanical conversion coefficient of the resonator in coupled vibration. From the above analysis, equations (41) and (42)

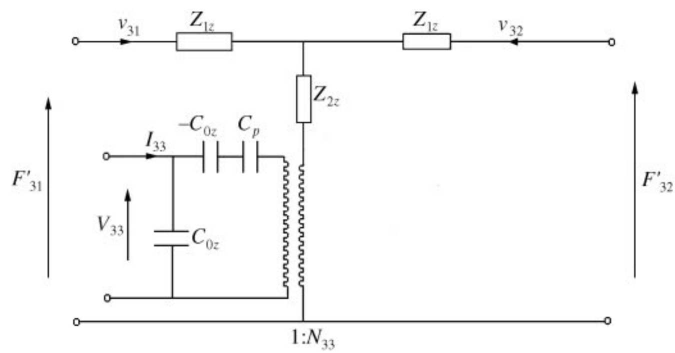


Figure 3. The equivalent circuit of the equivalent thickness vibration of the resonator in coupled vibration.

can be rewritten as

$$F'_{31} = \left(\frac{\rho S_z V_z}{\mathrm{j}\sin k_z l} - \frac{N_{33}^2}{\mathrm{j}\omega C_{0z}} + \frac{N_{33}^2}{\mathrm{j}\omega C_p}\right)(v_{31} + v_{32}) + \mathrm{j}\rho S_z V_z \tan\left(\frac{k_z l}{2}\right) v_{31} + N_{33} V_{33}, \tag{47}$$

$$F'_{32} = \left(\frac{\rho S_z V_z}{\mathrm{j}\sin k_z l} - \frac{N_{33}^2}{\mathrm{j}\omega C_{0z}} + \frac{N_{33}^2}{\mathrm{j}\omega C_p}\right)(v_{31} + v_{32}) + \mathrm{j}\rho S_z V_z \tan\left(\frac{k_z l}{2}\right) v_{32} + N_{33} V_{33}. \tag{48}$$

Here

$$C_p = \frac{\pi a^2 \varepsilon_{33}^{\mathrm{T}}}{l[1 + (d_{33} + d_{31}/n) d_{33} E_z/\varepsilon_{33}^{\mathrm{T}}]} \times \frac{(d_{33} + d_{31}/n)}{d_{31}/n}$$
$$= C_{0z}\left(1 + \frac{d_{33}}{d_{31}} n\right), F'_{31} = F_{31} + F_{3a},$$

$$F_{32} = F_{32} + F_{3a}, F_{3a} = \frac{\pi a^2 d_{31} E_z/n}{l[1 + (d_{33} + d_{31}/n) d_{33} E_z/\varepsilon_{33}^{\mathrm{T}}]} \times V_{33}, S_z = \pi a^2.$$

$C_p$ and $F_{3a}$ are caused by the coupling between the radial and thickness vibrations in the disk resonator. From equations (46)–(48), the equivalent circuit of the equivalent thickness vibration in coupled vibration can be obtained as shown in Figure 3. In the figure, $Z_{1z} = \mathrm{j}\rho S_z V_z \tan(k_z l/2)$, $Z_{2z} = \rho S_z V_z/\mathrm{j}\sin(k_z l)$. It can be seen that it is different from the traditional one-dimensional equivalent circuit of a slender piezoelectric ceramic rod and an additional capacitance is created. The additional capacitance has resulted from the coupling between the longitudinal and the radial vibrations in the resonator.

### 2.3. THE EQUIVALENT CIRCUIT OF THE RESONATOR IN COUPLED VIBRATION

In general cases, the resonator is excited by an external alternating electric signal. Let the voltage and current of the resonator in coupled vibration be $V_3$ and $I_3$.

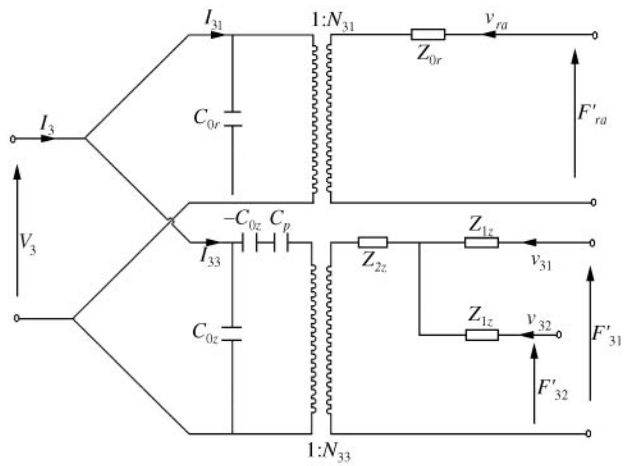


Figure 4. The equivalent circuit of the disk resonator in coupled vibration.

From the above analysis, the following relations can be obtained:

$$I_3 = I_{31} + I_{33}, \tag{49}$$

$$V_3 = V_{31} = V_{33}. \tag{50}$$

Combining the above analysis and results with equations (49) and (50), the equivalent circuit of the resonator in coupled vibration can be derived as shown in Figure 4.

## 3. THE RESONANCE FREQUENCY EQUATIONS OF THE DISK RESONATOR IN COUPLED VIBRATION

To analyze the frequency characteristics of the disk resonator in coupled vibration, the frequency equation of the resonator must be derived. From Figure 4, when the resonator vibrates freely, the external forces can be ignored. The admittance of the resonator in coupled vibration can be obtained:

$$Y_3 = \frac{I_3}{V_3} = \frac{I_{31} + I_{33}}{V_3} = Y_{31} + Y_{33}. \tag{51}$$

Here, $Y_{31}$ and $Y_{33}$ are the equivalent electric admittances of the resonator in equivalent radial and thickness vibrations respectively. Their expressions are as follows:

$$Y_{31} = \frac{I_{31}}{V_{31}} = \frac{j\omega\pi a^2 \varepsilon_{33}^{T}}{l} \times \left[1 - (K_p')^2 + (K_p')^2 \frac{(1 + v_{12}) J_1(k_r a)}{k_r a J_0(k_r a)(1 - v_{13} n) - J_1(k_r a)(1 - v_{12} - 2v_{13} n)}\right], \tag{52}$$

$$Y_{33} = \frac{I_{33}}{V_{33}} = \frac{\mathrm{j}\omega\pi a^2 \varepsilon_{33}^{\mathrm{T}}}{l} \times \frac{1 - (K'_{33})^2}{1 - (K'_{33})^2 \tan(k_z l/2)/(k_z l/2)}, \tag{53}$$

where $(K'_p)^2 = 2d_{31}(d_{31} + nd_{33})/[(s_{11}^E + s_{12}^E + 2s_{13}^E n)\varepsilon_{33}^{\mathrm{T}}]$. $K'_p$ is defined as the plane electro-mechanical coupling coefficient of the piezoelectric ceramic disk resonator in coupled vibration. It can be rewritten as

$$(K'_p)^2 = K_p^2 \times \frac{1 - n/\lambda_{31}}{1 - 2\nu_{13}n/(1 - \nu_{12})}, \tag{54}$$

where $K_p^2 = 2d_{31}^2/[\varepsilon_{33}^{\mathrm{T}}(s_{11}^E + s_{12}^E)]$, $K_p$ is the plane electro-mechanical coupling coefficient of a thin disk resonator in ideal radial vibration, $\lambda_{31} = -d_{31}/d_{33}$, $\nu_{12} = -s_{12}^E/s_{11}^E$, $\nu_{13} = -s_{13}^E/s_{11}^E$. $(K'_{33})^2 = d_{33}(d_{33} + d_{31}/n)/\varepsilon_{33}^{\mathrm{T}}(s_{33}^E + s_{13}^E/n)$, and $K'_{33}$ is the equivalent longitudinal electro-mechanical coupling coefficient of the resonator in coupled vibration. It can be rewritten as

$$(K'_{33})^2 = K_{33}^2 \times \frac{1 - \lambda_{31}/n}{1 - \nu_{31}/n}. \tag{55}$$

When the admittance of the resonator has a maximal value, the resonator will resonate. Therefore, the resonance frequency equations for the resonator in coupled vibration can be obtained from equations (52) and (53):

$$k_r a J_0(k_r a)(1 - \nu_{13}n) - J_1(k_r a)(1 - \nu_{12} - 2\nu_{13}n) = 0, \tag{56}$$

$$1 - (K'_{33})^2 \tan(k_z l/2)/(k_z l/2) = 0. \tag{57}$$

In equations (56) and (57), there are two unknowns: the mechanical coupling coefficient and the angular frequency. When the material parameters and the dimensions of the resonator are given, the resonance frequency of the resonator in coupled vibration can be computed from these two equations. However, since equations (56) and (57) are transcedental equations, it is impossible to find the analytic solutions. Therefore, numerical methods must be used. In solving these two transcedental equations, it can be found that for certain vibrational mode (for fundamental mode, the first roots of equations (56) and (57) are used), there exist two sets of solutions, i.e., two resonance frequencies can be found. Considering that the coupled vibration of the resonator includes two vibrational modes, it is obvious that these two frequencies are the resonance frequencies of the resonator in longitudinal and radial vibrations. One is the first longitudinal-dominating coupled mode, and the other is the first radial-dominating coupled mode. It can be seen that these two resonance frequencies are different from those calculated from the one-dimensional theory for the resonator in longitudinal or radial vibrational mode. The reason is that in this paper the interaction between the longitudinal and radial vibrations is considered. It can also be seen that when the geometrical dimensions of the resonator satisfy certain conditions (for example, when the thickness $l$ is much larger or smaller than its radius $a$), the interaction can be ignored, and the results from equations (56) and (57) are the same as those from one-dimensional theory.

For solving equations (56) and (57), the following procedures are used. First, an arbitrary value of the mechanical coupling coefficient is given, and then two frequencies can be found from equations (56) and (57). Second, the value of the mechanical coupling coefficient is changed until the two frequencies from the frequency equations (56) and (57) are equal to each other. In this case, the frequency and the mechanical coupling coefficient are the solutions of the frequency equations. By means of this method, the resonance frequencies corresponding to different vibrational modes of piezoelectric ceramic circular disk resonators in coupled vibration can be found.

## 4. EXPERIMENT

The resonance frequency of the piezoelectric ceramic disk resonator is measured to test and verify the proposed theory for the analysis of the disk resonator in coupled vibration. The experimental method used is the traditional transmission line method that is widely used in the performance measurement of the piezoelectric ceramic devices. The input sine electric signal applied to the resonator is small and so no non-linear effect is created in the resonator. The material of the piezoelectric ceramic disk resonator is an equivalent of PZT-4. In the computation of the resonance frequency of the resonator, the standard material parameters are used. The computed and measured results are listed in Table 1, where $f_r$ and $f_z$ are the computed resonance frequencies of the disk resonator in coupled fundamental vibrational mode. For comparison, the fundamental resonance frequencies $f_{1r}$ and $f_{1z}$ of the disk resonator in thickness extensional and radial vibrations computed from one-dimensional theory are also shown in Table 1; $f_{rm}$ and $f_{zm}$ are the measured results. It can be seen that the measured frequencies are in good agreement with the computed results, and the results from the theory of coupled vibration are in better agreement with the measured results than those from one-dimensional theory. From Table 1, it can be seen that the computed resonance frequency $f_r$ for the radial-dominating coupled mode is lower than the computed resonance frequency $f_{1r}$ for the plane radial vibration of the thin disk resonator, while the computed resonance frequency $f_z$ for the longitudinal-dominating coupled mode is higher than the computed resonance frequency $f_{1z}$ for the one-dimensional longitudinal vibration of a slender piezoelectric ceramic rod. This

Table 1

*The measured resonance frequencies of the piezoelectric ceramic disk resonators*

| 1 (mm) | $a$ (mm) | $f_{1r}$ (kHz) | $f_{1z}$ (kHz) | $f_r$ (kHz) | $f_z$ (kHz) | $f_{rm}$ (kHz) | $f_{zm}$ (kHz) |
|---|---|---|---|---|---|---|---|
| 5 | 30 | 38·24 | 402·25 | 38·08 | 415·27 | 37·01 | 419·22 |
| 6 | 30 | 38·24 | 335·19 | 38·04 | 346·26 | 36·95 | 341·62 |
| 8 | 30 | 38·24 | 251·39 | 37·90 | 260·51 | 36·63 | 264·33 |

can be explained as follows. First, for the plane radial vibration of a thin disk, its circumferential boundary is free from external force. When its thickness is increased, the coupled vibration is created, the energy for the longitudinal vibration is introduced, and the resonance frequency for the radial-dominating coupled vibration mode is decreased. Second, for the ideal thickness vibration of a think disk resonator, its circimferential boundary is clamped, the radial strain is zero, while the longitudinal strain has a certain value. This implies theoretically that the lateral dimension of the resonator is very large, and the mass of the resonator also has a maximal value. When the thickness of the resonator is increased, the coupled vibration is produced, and the condition of clamped boundary is no longer suitable for the resonator. Compared with the ideal thickness vibration of a thin disk resonator whose mass has a maximal value, the mass of the resonator in coupled vibration will decrease. Therefore, the resonance frequency for the longitudinal-dominating coupled mode is increased as compared with results of the one-dimensional theory.

In this paper, the fundamental vibrational mode of piezoelectric ceramic circular disk in coupled vibration is studied. The reason is that in most cases the fundamental mode of piezoelectric vibrators is widely used. It has high electro-mechanical coupling efficient, sensitivity, and low loss. As for the higher vibrational modes, the analysis is similar to that described in the above sections. For example, for the second vibrational mode, the second roots of equations (56) and (57) must first be found. Then the resonance frequency for the second vibrational mode of the disk resonator can be computed. However, the analysis for the high vibrational modes is complex. The reason is that the modal interaction must be considered. For example, for the second vibrational mode, the interaction between the radial vibration of order one and the thickness vibration of order two, the interaction between the thickness vibration of order one and the radial vibration of order two, and the interaction between the radial vibration of order two and the thickness vibration of order two must be analyzed at the same time.

As for the frequency error, it is considered that the following factors should be taken into account: (1) The standard material parameters is different from the practical values. An error of 3–5% can be caused by this factor. (2) To simplify the analysis, the mechanical coupling coefficient is considered as a constant. However, the mechanical coupling coefficient is different at different positions in the resonator. (3) The longitudinal and radial extensional vibrations in the resonator are assumed. However, when the disk resonator is a short cylinder or a think disk, shearing and other strains may exist in the resonator. (4) The analytical method presented in this paper is an approximate one.

## 5. CONCLUSIONS

In this paper, the coupled vibration of a piezoelectric ceramic disk resonator is studied. An approximate analytic method is developed to analyze the complex coupled vibration, and the equivalent circuit for the disk resonator in coupled vibration is obtained. From the above analysis, the following conclusions can be drawn:

(1) When the mechanical coupling coefficient is introduced, the complex coupled vibration of the disk resonator can be divided into two equivalent extensional vibrations: one is the longitudinal vibration, and the other is the plane radial vibration. However, these two vibrations are not independent of each other.
(2) The equivalent circuit for the disk resonator in coupled vibration is derived. An additional force is produced in the equivalent circuit, and this is different from the traditional result according to one-dimensional theory.
(3) The resonance frequency equations for the disk resonator in coupled vibration are obtained. Two resonance frequencies of the resonator in coupled vibration can be obtained from the theory of this paper in an analytical method. These two frequencies correspond to the coupled longitudinal and radial vibrations in the resonator.
(4) The coupled vibration of a disk resonator is complex. Apart from the fundamental mode and the higher modes, there exist vibrations in different directions and their interaction. For example, for the fundamental mode, there are two vibrations, one being the equivalent radial vibration, and the other the equivalent longitudinal vibration.
(5) The present method in this paper is an approximate method. It neglects the shearing and torsional strains. On the other hand, the mechanical coupling coefficient is considered as a constant for a certain vibrational mode.
(6) The method presented in this paper can be used to analyze and compute the resonance frequency of the piezoelectric ceramic disk resonators in coupled longitudinal and radial vibration. The coupled mode shapes cannot be calculated using this method. However, numerical methods such as the finite element method can be used to accomplish this task.

## REFERENCES


1. J. Van Randeraat and R. E. Setterington 1974 *Piezoelectric Ceramics*. London: Mullard Limited.
2. W. P. Mason 1964 *Physical Acoustics* 1, Part A. New York and London: Academic Press.
3. O. E. Mattiat 1971 *Ultrasonic Transducer Materials*. New York and London: Plenum Press.
4. E. A. G. Shaw 1956 *Journal of the Acoustical Society of America* **28**, 38–50. On the resonant vibrations of think Barium Titanate disks.
5. H. A. Kunkel, S. Locke and B. Pikeroen 1990 *IEEE Transactions on Ultrasonic Frequency Control* **37**, 316–327. Finite-element analysis of vibrational mode in piezoelectric ceramic disks.
6. P. Schnabel 1978 *IEEE Transactions on Sonics and Ultrasonics* **25**, 16–24. Dispersion of thickness vibrations of piezoelectric disk resonators.
7. G. Ningqun and P. Cawley 1992 *Journal of the Acoustical Society of America* **92**, 3377–3388. Measurement and prediction of the frequency spectrum of piezoelectric disks by modal analysis.
8. S. Ueha, S. Sakuma and E. Mori 1983 *Journal of the Acoustical Society of America* **73**, 1842–1847. Measurement of vibration velocity distributions and mode analysis in thick disks of $Pb(ZrTi)O_3$.
9. S. Y. Lin 1998 *Journal of Sound and Vibration* **218**, 205–217. Coupled vibration analysis of the piezoelectric ceramic disk resonators.